Kolmogorov's strong law of large numbers in game-theoretic probability: Reality's side

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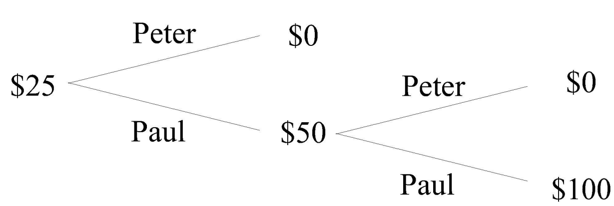

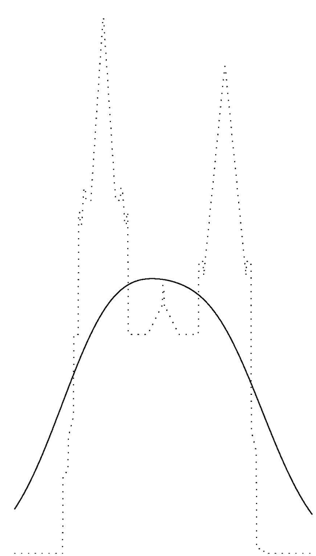

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1 Statement

This note describes a simple explicit strategy for Reality whose existence is asserted in Theorem 4.1 (part 2) of [2] (p. 80). We will be using the notation of [2] complemented by $S_n := x_1 + \cdots + x_n$; without loss of generality we can assume that $m_n = 0$ for all n. Namely, we construct an explicit strategy for Reality that guarantees

$$\mathcal{K}_n \text{ is bounded and } \left(\sum_{n=1}^{\infty} \frac{v_n}{n^2} = \infty \Longrightarrow \left(\frac{S_n}{n} \not\to 0\right)\right) \tag{1}$$

provided Skeptic satisfies his collateral duty (keeping $\mathcal{K}_n$ non-negative).

For much more advanced results, see Theorems 4.12 and 5.10 of [1]. The main advantage of this note is its brevity.

2 Reality's strategy and proof

With Skeptic's move (M_n, V_n) we associate the function $f_n(x) := M_n x + V_n(x^2 - v_n)$; the increase in his capital will be $f_n(x_n)$. We will assume that $M_n = 0$: it will be clear that Reality can exploit $M_n \neq 0$ by choosing the sign of x_n. Our argument will also be applicable to the modified protocol of unbounded forecasting in which Skeptic can choose any $V_n \in \mathbb{R}$: Reality can easily win when $V_n < 0$ by choosing $|x_n|$ large enough.

This is Reality's strategy:

1. Keep setting $x_n := 0$ until Skeptic chooses a move for which $\mathcal{K}_{n-1} + f_n(n) \leq 1$.

2. When Skeptic chooses such a move, set $x_n := n$ or $x_n := -n$. Go to 1.

Notice that Skeptic's capital is guaranteed to be bounded by 1, and so Reality satisfies her collateral duty. Consider two cases:

- Suppose that item 2 is reached infinitely often. Since $S_n/n \not\to 0$ as $n \to \infty$, (1) will be satisfied.

- Now suppose Skeptic reaches item 2 only finitely many times. From some n on, we will have $f_n(n) > 1 - \mathcal{K}_{n-1}$, i.e., $V_n(n^2 - v_n) > 1 - \mathcal{K}_{n-1}$, which implies $V_n > n^{-2}(1 - \mathcal{K}_{n-1})$. Therefore, from some n on Skeptic

will lose at least $|f_n(0)| = V_n v_n \ge v_n n^{-2}(1 - \mathcal{K}_{n-1})$. Suppose, without loss of generality, $\sum_n v_n n^{-2} = \infty$ (otherwise (1) is satisfied). Since the sequence $1-\mathcal{K}_{n-1}$ is increasing from some n on and there are arbitrarily large n for which $v_n > 0$ and hence $1 - \mathcal{K}_n > 1 - \mathcal{K}_{n-1} \ge 0$, Skeptic will eventually become bankrupt.

Acknowledgement

This note was prompted by Akimichi Takemura's question in 2006; I am also grateful to him for several useful discussions.

References

[1] Kenshi Miyabe and Akimichi Takemura. Convergence of random series and the rate of convergence of the strong law of large numbers in game-theoretic probability. *Stochastic Processes and their Applications*, 122:1–30, 2012.

[2] Glenn Shafer and Vladimir Vovk. *Probability and Finance: It's Only a Game!* Wiley, New York, 2001.